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Strong Rabin numbers of folded hypercubes

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Abstract

The strong Rabin number of a network W of connectivity k is the minimum l so that for any $k+1$ nodes $s, d_1, d_2, \ldots, d_k$ of W, there exist k node-disjoint paths from s to $d_1, d_2, \ldots, d_k$, respectively, whose maximal length is not greater than l, where $s \notin \{d_1, d_2, \ldots, d_k\}$ and $d_1, d_2, \ldots, d_k$ are not necessarily distinct. In this paper, we show that the strong Rabin number of a k-dimensional folded hypercube is $\lceil k/2 \rceil + 1$, where $\lceil k/2 \rceil$ is the diameter of the k-dimensional folded hypercube. Each node-disjoint path we obtain has length not greater than the distance between the two end nodes plus two. This paper solves an open problem raised by Liaw and Chang.

Keywords: Folded hypercube; Hypercube; Node-disjoint paths; Optimization problem; Strong Rabin number

1. Introduction

A k-dimensional hypercube (abbreviated to a k-cube) [18] consists of 2^k nodes whose names (or labels) are the 2^k binary strings of length k. Two nodes are connected with a link if and only if they differ by exactly one bit. The diameter of a k-cube is k. On the other hand, a k-dimensional folded hypercube (abbreviated to a k-fcube) [6] is basically a k-cube augmented with 2^{k-1} complement links. Each *complement link* connects two nodes whose names are the complements of each other. Fig. 1 shows the structure of a 3-fcube, where $(000,\ 111)$, $(001,\ 110)$, $(010,\ 101)$ and $(011,\ 100)$ are the four complement links.

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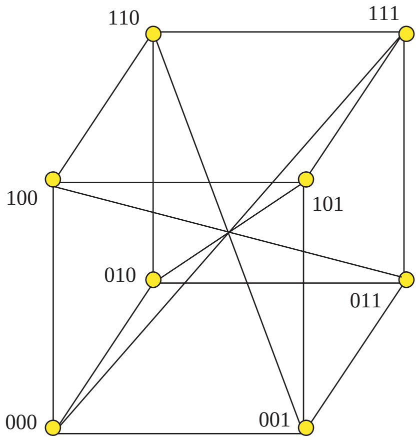

Fig. 1. The structure of a 3-fcube.

Due to the complement links, the diameter of a k-fcube is reduced to $\lceil k/2 \rceil$. A k-cube and a k-fcube have connectivities k and $k+1$, respectively. The *connectivity* of an interconnection network (network for short) is the minimum number of nodes whose removal can cause the network disconnected or trivial [2].

Routing is a process of transmitting messages among nodes and its efficiency is crucial to the performance of a network. Efficient and reliable routing can be achieved by using internally node-disjoint paths (disjoint paths for short), because they can be used to avoid congestion, accelerate the transmission rate, and provide alternative transmission routes. In order to reduce the transmission latency, disjoint paths with maximal length minimized are desired. For this purpose, the concept of Rabin number was introduced (see [5]). The *Rabin number* of a network W was defined to be the minimal l so that for any $k+1$ distinct nodes of W, there exist k disjoint paths from one (source) node to the other k (destination) nodes, respectively, whose maximal length is not greater than l, where k is the connectivity of W. The Rabin number of W represents the maximum transmission latency when transmitting messages from the source node to the k destination nodes in parallel.

According to Theorem 2.6 in [1], there exist k disjoint paths from one node to another k distinct nodes in W. Rabin numbers were obtained for some particular networks, e.g., hypercube [17], folded hypercube [11,12], pyramid [3], butterfly network [14,15], circulant network [13], d-ary cube [13], generalized hypercube [13], and WK-recursive network [13]. An upper bound on the Rabin number of the star network was proposed in [4]. It was shown in [10] that the Rabin number problem for a general graph is NP-hard.

Recently, Liaw and Chang [13] generalized the definition of the Rabin number so that the destination nodes are not necessarily distinct. The resulting Rabin number was called the *strong Rabin number*. In [13], strong Rabin numbers of hypercube, circulant network, d-ary cube, generalized hypercube and WK- recursive network were computed. The strong Rabin number of the folded hypercube was left unsolved in [13]. In [7], Gao and Hsu suggested an upper bound on the star diameter of Cayley graphs, where the star diameter has the same definition as the strong Rabin number.

There are three categories of disjoint paths: one-to-one, one-to-many, and many-to-many. Disjoint paths constructed for the container problem [9,16] belong to the one-to-one category, while those constructed for the Rabin number problem belong to the one-to-many category. The strong Rabin number problem, which combines the features of the two problems, can construct one-to-one and one-to-many disjoint paths in two extreme cases. On the other hand, multicasting on a network also requires one-to-many disjoint paths whose maximal length is minimized. For a constrained multicasting in which some destination nodes require larger data rate and higher transmission reliability than the others, the strong Rabin number problem can serve the purpose.

For the same network, the strong Rabin number is greater than or equal to the Rabin number. Recently, the Rabin number of a k-fcube was computed in [11,12], which is equal to $\lceil k/2 \rceil + 1$. In this paper, we show that the strong Rabin number of a k-fcube is also $\lceil k/2 \rceil + 1$, which solves the open problem proposed in [13]. The resulting disjoint paths have maximal length not greater than the maximal distance from the source node to the destination nodes plus two.

In the next section, two related works, i.e., [11,12], are briefly reviewed. In Section 3, two procedures named Paths3 and Paths4 that can produce $k+1$ disjoint paths from one node to another (not necessarily distinct) $k+1$ nodes, respectively, in a k-fcube are presented. It is shown in Section 4 that the $k+1$ disjoint paths have maximal length not greater than $\lceil k/2 \rceil + 1$. Consequently, the strong Rabin number of a k-fcube is equal to $\lceil k/2 \rceil + 1$. In Section 5, this paper concludes with some remarks.

2. Preliminaries

In this section we first review routing functions. It was shown in [11,12] that routing functions can be used to derive disjoint paths in the hypercube and folded hypercube. Two procedures named Paths1 and Paths2 were proposed in [11,12]. Paths1 can produce m disjoint paths from one node to another m (not necessarily distinct) nodes in a k-cube, where $m \leqslant k$. The m disjoint paths have maximal length not greater than $k+1$, where $k+1$ is the Rabin number of a k-cube [5]. Paths2 can produce $k+1$ disjoint paths from one node to another $k+1$ distinct nodes in a k-fcube. The $k+1$ disjoint paths have maximal length not greater than $\lceil k/2 \rceil + 1$, where $\lceil k/2 \rceil + 1$ is the Rabin number of a k-fcube [12].

2.1. Routing functions

Suppose that s is the source node and $d_1, d_2, \ldots, d_m$ are m (not necessarily distinct) destination nodes in a k-cube, where $m \leqslant k$ and $s \neq d_i$ for all $1 \leqslant i \leqslant m$. Since the hypercube is node symmetric, we assume $s = \overbrace{00\ldots0}^{k} = 0^k$ without losing generality. Let $D = \{d_1, d_2, \ldots, d_m\}$ be a multiset and $I = \{n_1, n_2, \ldots, n_m\}$ be a set of m distinct integers ranging from 1 to k ($n_1, n_2, \ldots, n_m$ denote m distinct dimensions of a k-cube). A *multiset* is a collection of elements in which multiple occurrences of the same element are allowed. A one-to-one mapping Φ from D to I, which can be used to derive disjoint paths from s to $d_1, d_2, \ldots, d_m$, is a *routing function* for a k-cube.

An intuitive meaning of $\Phi(d_i) = n_t$ is to assign e_{n_t} as the immediate successor of s in the path to d_i, where $e_{n_t} = 0^{n_t-1}10^{k-n_t}$. Throughout this paper, we use $d_{i,j}$ to denote the jth bit (from the left) of d_i, where $1 \leqslant j \leqslant k$. That is, $d_i = d_{i,1}d_{i,2} \ldots d_{i,k}$ is assumed. Since we would like to have each path as short as possible, a Φ with $d_{i,\Phi(d_i)} = 1$ for all $1 \leqslant i \leqslant m$ is preferred. However, such a Φ does not always exist for arbitrary D and I.

We use $|d_i|$ to denote the number of bits 1 contained in d_i (i.e., the distance from s to d_i), and v_j to denote the number of d_i's in D so that $|d_i| = j$ and $d_{i,\Phi(d_i)} = 0$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$. For example, if $m = k = 5$, $(d_1, d_2, d_3, d_4, d_5) = (00011,\ 00101,\ 00111,\ 00110,\ 11100)$ and $(\Phi(d_1), \Phi(d_2), \Phi(d_3), \Phi(d_4), \Phi(d_5)) = (5, 3, 2, 4, 1)$, then $(v_1, v_2, v_3, v_4, v_5) = (0, 0, 1, 0, 0)$. We have $(v_1, v_2, \ldots, v_k) < (v'_1, v'_2, \ldots, v'_k)$ if there is some $1 \leqslant l \leqslant k$ such that $v_i = v'_i$ for all $1 \leqslant i < l$ and $v_l < v'_l$. Let $V_\Phi = (v_1, v_2, \ldots, v_k)$. We say that Φ is *minimal* if $V_\Phi \leqslant V_{\Phi'}$ for any $\Phi' : D \to I$. A minimal Φ can be determined in $O(k^3)$ time (see [11]).

Intuitively, a minimal Φ prefers $d_{p,\Phi(d_p)} = 1$ to $d_{q,\Phi(d_q)} = 1$ for $|d_p| < |d_q|$. The purpose of a minimal Φ is to derive shorter disjoint paths in a k-cube. In [11], by the aid of a minimal Φ, m disjoint paths from s to $d_1, d_2, \ldots, d_m$, which are shortest or second shortest, were obtained.

2.2. Procedures Paths1($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$) and Paths2($s, d_1, d_2, \ldots, d_{k+1}$)

When Φ is minimal, Paths1($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$) can produce m disjoint paths from s to $d_1, d_2, \ldots, d_m$, respectively, whose maximal length is not greater than $\min\{\mathrm{dis}_{\max} + 2, k + 1\}$, where $\mathrm{dis}_{\max}$ denotes the maximal distance from s to $d_1, d_2, \ldots, d_m$. The execution of Paths1($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$) invokes another procedure, i.e., Paths0($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$), which can produce m disjoint shortest paths connecting $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$. By augmenting these m paths with links $(s, e_{n_1}), (s, e_{n_2}), \ldots, (s, e_{n_m})$, m disjoint paths from s to $d_1, d_2, \ldots, d_m$ can be obtained. The executions of Paths0($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$) and Paths1($\Phi, m, k, \{d_1, d_2, \ldots, d_m\}, \{n_1, n_2, \ldots, n_m\}$) were detailed in [11]. The following lemma was shown in [11].

Lemma 1 (*Lai [11]*). *Paths1($\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\}$) can produce k disjoint paths from s to $d_1, d_2, \ldots, d_k$, respectively, where $\Phi\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. Besides, for all $1 \leqslant i \leqslant k$, the path to d_i has length $|d_i|$ if $d_{i,\Phi(d_i)} = 1$, and length $|d_i| + 2$ if $d_{i,\Phi(d_i)} = 0$. If the path to d_i has length $|d_i| + 2$, then $|t_i| = |d_i| + 1$, where t_i is the immediate predecessor of d_i in the path to d_i.*

In [8], Gao and Novick also constructed k disjoint paths from s to $d_1, d_2, \ldots, d_k$ whose total length is minimized. Besides, the path from s to d_i has length at most $|d_i| + 2$.

On the other hand, when $s, d_1, d_2, \ldots, d_{k+1}$ are all distinct, Paths2($s, d_1, d_2, \ldots, d_{k+1}$) can produce $k+1$ disjoint paths, denoted by $Q_1, Q_2, \ldots, Q_{k+1}$, from s to $d_1, d_2, \ldots, d_{k+1}$, respectively, in a k-fcube whose maximal length is not greater than $\lceil k/2 \rceil + 1$. Since the folded hypercube is node symmetric, we assume $s = 0^k$ without losing generality. The execution of Paths2($s, d_1, d_2, \ldots, d_{k+1}$) is outlined as follows. Suppose $|d_{k+1}| \geqslant |d_i|$ for all $1 \leqslant i \leqslant k$. First, k disjoint paths $Q'_1, Q'_2, \ldots, Q'_k$ are obtained by invoking Paths1, where each $Q'_i (1 \leqslant i \leqslant k)$ ends at d_i or $\overline{d_i}$ (binary complement of d_i) or an adjacent node of $\overline{d_i}$ (in a k-cube). Then, each Q_i is obtained from Q'_i by adding at most two links. Finally, Q_{k+1} is constructed as the combination of $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} (in a k-fcube). If Q_{k+1} is not disjoint with $Q_1, Q_2, \ldots, Q_k$, then some modifications are made so that they become disjoint.

3. Constructing disjoint paths in folded hypercubes

Recall that Paths2$(s, d_1, d_2, \ldots, d_{k+1})$ require $d_1, d_2, \ldots, d_{k+1}$ all distinct. In this section, allowing $d_1, d_2, \ldots, d_{k+1}$ not necessarily distinct, two procedures, i.e., Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ and Paths4$(s, d_1, d_2, \ldots, d_{k+1})$, are proposed that can construct $k+1$ disjoint paths from s to $d_1, d_2, \ldots, d_{k+1}$, respectively, in a k-fcube. Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ assumes that at most one of $d_1, d_2, \ldots, d_{k+1}$ is 1^k, and Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ assumes that two or more of $d_1, d_2, \ldots, d_{k+1}$ are 1^k. The $k+1$ disjoint paths have maximal length at most $\lceil k/2 \rceil + 1$, which implies that the strong Rabin number of a k-fcube is equal to $\lceil k/2 \rceil + 1$.

3.1. *Procedure Paths3*$(s, d_1, d_2, \ldots, d_{k+1})$

We use $Q_1, Q_2, \ldots, Q_{k+1}$ to denote the $k+1$ paths produced by Paths3$(s, d_1, d_2, \ldots, d_{k+1})$, where $Q_i (1 \leqslant i \leqslant k+1)$ is the path from s to d_i. Essentially, Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ is an extension of Paths2$(s, d_1, d_2, \ldots, d_{k+1})$, which reflects the change that $d_1, d_2, \ldots, d_{k+1}$ are not necessarily distinct. A formal description of Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ is shown below. Most of the statements in Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ are the same as those in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Statement numbers with * (e.g., 6*, 9*, and 17*) denote statements that are modifications of the corresponding ones in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Statement numbers with A (e.g., A1, A2, and A3) denote statements that are new with respect to Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Statement numbers without $*$ and A denote duplicates from Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Statements in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$ that are different from those in Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ are collected in Appendix. We say that a node x is *contained internally* in Q_i, denoted by $x \hat{\in} Q_i$, if $x \in Q_i - \{s, d_i\}$, where $1 \leqslant i \leqslant k+1$. Similarly, we use $x \hat{\notin} Q_i$ to denote that x is not contained internally in Q_i.

Procedure Paths3$(s, d_1, d_2, \ldots, d_{k+1})$.

(1) *Step* 1: Determine d_t so that $|d_t| \geqslant |d_i|$ for all $1 \leqslant i \leqslant k+1$, where $1 \leqslant t \leqslant k+1$. Without loss of generality, we assume $t = k+1$.
(2) *Step* 2: Construct $Q_1, Q_2, \ldots, Q_{k+1}$ according to the following five cases:
(3) *Case* 1: $|d_{k+1}| \leqslant \lceil k/2 \rceil - 2$.
(4) /* Suppose that $\Phi : \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. */
(5) Construct $Q_1, Q_2, \ldots, Q_k$ in a k-cube by invoking Paths1$(\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$.
(6*) If $d_{k+1} \hat{\in} Q_l$ for some $1 \leqslant l \leqslant k$, then {
(7) Construct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.
(8) Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$.}

(9*) If $d_{k+1} \hat{\notin} Q_i$ for all $1 \leqslant i \leqslant k$, then
(10) Construct Q_{k+1} as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_{k+1}}$ in a k-cube, and a link $(\overline{d_{k+1}}, d_{k+1})$.
(11) *Case* 2: $|d_{k+1}| = \lceil k/2 \rceil - 1$.
(12) If k is even, then
(13) Construct $Q_1, Q_2, \ldots, Q_{k+1}$ all the same as in Case 1.
(14) If k is odd, then {
(15) /* Suppose that $\Phi : \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. */
(16) Construct $Q_1, Q_2, \ldots, Q_k$ in a k-cube by invoking Paths1($\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\}$).
(17*) If $d_{k+1} \hat{\in} Q_l$ for some $1 \leqslant l \leqslant k$, then {
(18) Construct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.
(19) If d_{k+1} is the immediate predecessor of d_l in Q_l, then
(20) Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$.
(21) If d_{k+1} is not the immediate predecessor of d_l in Q_l, then
(22) /* Suppose that x is the immediate predecessor of d_l in Q_l. */
(23) Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to x in a k-cube, and a link (x, d_l).}
(24*) If $d_{k+1} \hat{\notin} Q_i$ for all $1 \leqslant i \leqslant k$, then {
(25) Construct Q_{k+1} as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube.
(26*) If Q_{k+1} intersects with Q_h for some $1 \leqslant h \leqslant k$ at a node $y \notin \{s, d_{k+1}\}$, then {
(27) Reconstruct Q_{k+1} as the combination of the subpath of Q_h from s to y and a shortest path from y to d_{k+1}.
(28) /* Suppose that z is an adjacent node of d_h so that $|z| = |d_h| + 1$ and $z \notin Q_i$ for all $1 \leqslant i \leqslant k$. */
(29) Reconstruct Q_h as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to z in a k-cube, and a link (z, d_h).}}}
(30) *Case* 3: $|d_{k+1}| = \lceil k/2 \rceil$.
(31) /* Suppose that $\Phi : \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_r\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_k\}$ so that for all $1 \leqslant i \leqslant k$, $d_i \in \{d_1, d_2, \ldots, d_r\}$ if $d_{i,\Phi(d_i)} = 0$ and $|d_i| = \lceil k/2 \rceil$, and $d_i \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ else, where $0 \leqslant r \leqslant k$. Define a minimal routing function $\Omega : \{\overline{d_1}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ as follows: $\Omega(\overline{d_i}) = \Phi(d_i)$ for all $1 \leqslant i \leqslant r$ and $\Omega(d_j) = \Omega(d_j)$ for all $r + 1 \leqslant j \leqslant k$. */
(32) Construct $Q'_1, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$ in a k-cube by invoking Paths1($\Omega, k, k, \{\overline{d_1}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}, \{1, 2, \ldots, k\}$).
/* Assume that Q'_i ends at $\overline{d_i}$ for all $1 \leqslant i \leqslant r$. */
(33) For $i = 1, 2, \ldots, r$, {
(34) /* Suppose that f_i is the immediate predecessor of $\overline{d_i}$ in Q'_i. */
(35*) If $\overline{d_i} \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_{k+1}\}$, then
(36) Construct Q_i as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f_i})$ and $(\overline{f_i}, d_i)$.
(37*) If $\overline{d_i} \notin \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_{k+1}\}$, then
(38) Construct Q_i as the combination of Q'_i and a link $(\overline{d_i}, d_i)$.}
(39) Construct Q_{k+1} as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube.
(40*) If Q_{k+1} intersects with Q_t for some $1 \leqslant t \leqslant r$ at a node $x \notin \{s, d_{k+1}\}$, then {
(A1) If $\overline{d_t} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then {
(41*) /* Suppose that $\overline{d_t} = d_m$ and g is the immediate predecessor of d_m in Q_m, where $r + 1 \leqslant m \leqslant k$. */

(42) Reconstruct Q_t as the combination of the subpath of Q_m from s to g and two links $(g, \bar{g})$ and $(\bar{g}, d_t)$.

(43) Reconstruct Q_m as the combination of the subpath of (old) Q_t from s to $\bar{x}$ and a shortest path from $\bar{x}$ to d_m.}

(A2) If $\overline{d_t} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then

(A3) Reconstruct Q_t as the combination of the subpath of (old) Q_t from s to $\bar{x}$, a shortest path from $\bar{x}$ to $\overline{d_t}$, and a link $(\overline{d_t}, d_t)$. }

(44*) If $d_{k+1} \hat{\in} Q_l$ for some $r+1 \leqslant l \leqslant k$, then {

(45) Reconstruct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.

(46) If k is even, then

(47) Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$.

(48) If k is odd, then {

(49) /* Suppose that y is an adjacent node of d_l so that $|y| = |d_l| + 1$ and $y \notin Q_i$ for all $1 \leqslant i \leqslant k$. */

(50) Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to y in a k-cube, and a link (y, d_l).

(51) If Q_l intersects with Q_v for some $1 \leqslant v \leqslant r$ at a node $z (\neq s)$, then {

(A4) If $\overline{d_v} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then {

(52*) /* Suppose that $\overline{d_v} = d_w$ and p is the immediate predecessor of d_w in Q_w, where $r+1 \leqslant w \leqslant k$ and $w \neq l$. */

(53) Reconstruct Q_v as the combination of the subpath of Q_w from s to p and two links $(p, \bar{p})$ and $(\bar{p}, d_v)$.

(54) Reconstruct Q_w as the combination of the subpath of (old) Q_v from s to $\bar{z}$ and a shortest path from $\bar{z}$ to d_w. }

(A5) If $\overline{d_v} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then {

(A6) Reconstruct Q_v as the combination of the subpath of (old) Q_v from s to $\bar{z}$, a shortest path from $\bar{z}$ to $\overline{d_v}$, and a link $(\overline{d_v}, d_v)$.

(A7) If $\overline{d_v} \in Q_u$ for some $1 \leqslant u \leqslant r$ and $u \neq v$, then

(A8) /* Suppose that q is the immediate predecessor of $\overline{d_v}$ in Q_u. */

(A9) Reconstruct Q_u as the combination of the subpath of (old) Q_u from s to q, a link $(q, \bar{q})$, and a shortest path from $\bar{q}$ to d_u. }}}}

(55) *Case* 4: $|d_{k+1}| = \lceil k/2 \rceil + 1$.

(56*) /* Suppose that $\Phi : \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_r\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_k\}$ so that for all $1 \leqslant i \leqslant k$, $d_i \in \{d_1, d_2, \ldots, d_r\}$ if $d_{i,\Phi(d_i)} = 0$ and $|d_i| = \lceil k/2 \rceil + 1$, and $d_i \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ else, where $0 \leqslant r \leqslant k$. Initially, let $D = \{\overline{d_1}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}$ and define a minimal routing function $\Psi : D \to \{1, 2, \ldots, k\}$ as follows: $\Psi(\overline{d_i}) = \Phi(d_i)$ for all $1 \leqslant i \leqslant r$ and $\Psi(d_t) = \Phi(d_t)$ for all $r+1 \leqslant t \leqslant k$. Then, examine $d_{r+1}, d_{r+2}, \ldots, d_k$, sequentially. Whenever there is a $d_l \in D$, $r+1 \leqslant l \leqslant k$, so that $d_{l,\Psi(d_l)} = 0$ and $|d_l| = \lceil k/2 \rceil$, D and Ψ need to be modified as follows. If $\overline{d_l} \notin D$, then replace d_l with $\overline{d_l}$ in D and assign $\Psi(\overline{d_l})$ with $\Psi(d_l)$. Otherwise ($\overline{d_l} \in D$), let $p_l = p_{l,1} p_{l,2} \cdots p_{l,k}$ be an adjacent node of $\overline{d_l}$ so that $|p_l| = |\overline{d_l}| - 1$, $p_l \notin D$, and $\overline{p_l} \notin D \cup \{d_{k+1}\}$. First, swap $\Psi(d_l)$ and $\Psi(\overline{d_w})$ (or swap $\Psi(d_l)$ and $\Psi(d_h)$), if $p_{l,\Psi(d_l)} = 0$ and $\overline{d_l} = \overline{d_w}$ for some $r+1 \leqslant w \leqslant k$ (or $\overline{d_l} = d_h$ for some $r+1 \leqslant h \leqslant k$). Then, replace d_l with p_l in D, and assign $\Psi(p_l)$ with $\Psi(d_l)$. Without loss of generality, assume $D = \{\overline{d_1}, \ldots, \overline{d_u}, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_k\}$ finally, where $r \leqslant u \leqslant v \leqslant k$. */

(57) Construct $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ in a k-cube by invoking Paths1 $(\Psi, k, k, D, \{1, 2, \ldots, k\})$.

(58) /* Assume that Q'_j ends at $\overline{d_j}$ for all $1 \leqslant j \leqslant u$ and Q'_t ends at p_t for all $u+1 \leqslant t \leqslant v$. */

(59) Construct Q_{k+1} as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube.

(60) For $i = 1, 2, \ldots, r$, {

(61) /* We use f_i to denote the immediate predecessor of $\overline{d_i}$ in Q'_i. */

(62*) If $\overline{d_i} \in \{d_{v+1}, d_{v+2}, \ldots, d_k\}$, then {

(A10) /* Suppose $\overline{d_i} = d_w$, where $v+1 \leqslant w \leqslant k$. */

(63) If $\overline{f_i} \in Q_{k+1}$, then

(64) Swap Q'_i and Q_w.

(65) /* Now f_i denotes the immediate predecessor of $\overline{d_i}$ in (new) Q'_i. */

(66) Construct Q_i as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f_i})$ and $(\overline{f_i}, d_i)$. }

(A11) If $\overline{d_i} \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ but $\overline{d_i} \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}\}$, then {

(A12) If $\overline{f_i} \in Q_{k+1}$, then

(A13) Construct Q_i as the combination of Q'_i and a link $(\overline{d_i}, d_i)$.

(A14) If $\overline{f_i} \notin Q_{k+1}$, then

(A15) Construct Q_i as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f_i})$ and $(\overline{f_i}, d_i)$. }

(67*) If $\overline{d_i} \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\} \cup \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}\}$, then

(68) Construct Q_i as the combination of Q'_i and a link $(\overline{d_i}, d_i)$. }

(69) For $j = r+1, r+2, \ldots, u$,

(70) Construct Q_j as the combination of Q'_j and a link $(\overline{d_j}, d_j)$.

(71) For $t = u+1, u+2, \ldots, v$,

(72) Construct Q_t as the combination of Q'_t and two links $(p_t, \overline{p_t})$ and $(\overline{p_t}, d_t)$.

(73) *Case* 5: $|d_{k+1}| \geqslant \lceil k/2 \rceil + 2$.

(74*) /* Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_{u'}\}$ and $\{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ so that for all $1 \leqslant i \leqslant k$, $d_i \in \{d_1, d_2, \ldots, d_{u'}\}$ if $|d_i| \geqslant \lceil k/2 \rceil + 2$, and $d_i \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ else, where $0 \leqslant u' \leqslant k$. Initially, let $D = \{\overline{d_1}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ and determine a minimal routing function $\Omega : D \to \{1, 2, \ldots, k\}$. Then, examine $\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{u'}}$, sequentially. Whenever there is a $\overline{d_h} \in D$, $1 \leqslant h \leqslant u'$, so that $|\overline{d_h}| = \lfloor k/2 \rfloor - 2$ and at least one of the following three conditions holds: (1) ($\overline{d_h} = d_l$ for some $u'+1 \leqslant l \leqslant k$ or ($\overline{d_h} = \overline{d_\theta} \in D$ for some $1 \leqslant \theta \leqslant u'$ and $\theta \neq h$)) and $\overline{d_{h,\Omega(\overline{d_h})}} = 0$, (2) $\overline{d_h} = d_l$ for some $u'+1 \leqslant l \leqslant k$, and $\overline{d_{h,\Omega(d_l)}} = 0$, and (3) $\overline{d_h} = \overline{d_\theta} \in D$, and $\overline{d_{h,\Omega(\overline{d_\theta})}} = 0$ for some $1 \leqslant \theta \leqslant u'$ and $\theta \neq h$, we need to modify D and Ω as follows. First, if $\overline{d_{h,\Omega(\overline{d_h})}} = 1$, then swap $\Omega(\overline{d_h})$ and $\Omega(d_l)$ if the condition (2) holds, and swap $\Omega(\overline{d_h})$ and $\Omega(\overline{d_\theta})$ else. Let $q_h = q_{h,1} q_{h,2} \ldots q_{h,k}$ be an adjacent node of $\overline{d_h}$ so that $q_h \notin D$, $\overline{q_h} \notin D$, and for some $w \in \{1, 2, \ldots, k\} - \{\Omega(z) | z = z_1 z_2 \ldots z_k \in D, z_{\Omega(z)} = 1$, and $|z| \leqslant \lfloor k/2 \rfloor - 1\}$, $q_{h,w} = 1$, $\overline{d_{h,w}} = 0$, and $q_{h,i} = \overline{d_{h,i}}$ for all $1 \leqslant i \leqslant k$ and $i \neq w$. Then, replace $\overline{d_h}$ with q_h in D, and determine a new minimal routing function Ω arbitrarily. Without loss of generality, assume $D = \{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ finally, where $0 \leqslant r' \leqslant u'$. */

(75) Construct $k+1$ disjoint paths, denoted by $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$, from s to $q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_{k+1}$, respectively, by the construction method of Case 4 with an initial minimal Ω and destination nodes $q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_{k+1}$ (i.e., substituting $\Omega, q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}$ for $\Phi, d_1, d_2, \ldots, d_{u'}$ in Case 4, respectively).

(76) /* Assume that Q'_i ends at q_i for all $1 \leqslant i \leqslant r'$ and Q'_j ends at $\overline{d_j}$ for all $r'+1 \leqslant j \leqslant u'$. */

(77) For $i = 1, 2, \ldots, r'$,

(78) Construct Q_i as the combination of Q'_i and two links $(q_i, \overline{q_i})$ and $(\overline{q_i}, d_i)$.

(79) For $j = r' + 1, r' + 2, \ldots, u'$, {
(80) /* We use g_j to denote the immediate predecessor of $\overline{d_j}$ in Q'_j. */
(81*) If $\overline{d_j} \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$, then {
(A16) /* Suppose $\overline{d_j} = d_t$, where $u' + 1 \leqslant t \leqslant k$. */
(82) If $\overline{g_j} \in Q_{k+1}$, then
(83) Swap Q'_j and Q_t.
(84) /* Now g_j denotes the immediate predecessor of $\overline{d_j}$ in (new) Q'_j. */
(85) Construct Q_j as the combination of the subpath of Q'_j from s to g_j and two links $(g_j, \overline{g_j})$ and $(\overline{g_j}, d_j)$. }
(A17) If $\overline{d_j} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ but $\overline{d_j} \in \{\overline{d_{r'+1}}, \ldots, \overline{d_{j-1}}, \overline{d_{j+1}}, \ldots, \overline{d_{u'}}\}$, then {
(A18) If $\overline{g_j} \in Q_{k+1}$, then
(A19) Construct Q_j as the combination of Q'_j and a link $(\overline{d_j}, d_j)$.
(A20) If $\overline{g_j} \notin Q_{k+1}$, then
(A21) Construct Q_j as the combination of the subpath of Q'_j from s to g_j and two links $(g_j, \overline{g_j})$ and $(\overline{g_j}, d_j)$. }
(86*) If $\overline{d_j} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\} \cup \{\overline{d_{r'+1}}, \ldots, \overline{d_{j-1}}, \overline{d_{j+1}}, \ldots, \overline{d_{u'}}\}$, then
(87) Construct Q_j as the combination of Q'_j and a link $(\overline{d_j}, d_j)$. }

Step 1 has $|d_{k+1}| \geqslant |d_i|$ for all $1 \leqslant i \leqslant k$ in order to minimize the length of Q_{k+1} (Q_{k+1} is constructed as the combination of $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-fcube in Step 2). Step 2 constructs $Q_1, Q_2, \ldots, Q_{k+1}$ according to five cases, i.e., $|d_{k+1}| \leqslant \lceil k/2 \rceil - 2$, $|d_{k+1}| = \lceil k/2 \rceil - 1$, $|d_{k+1}| = \lceil k/2 \rceil$, $|d_{k+1}| = \lceil k/2 \rceil + 1$, and $|d_{k+1}| \geqslant \lceil k/2 \rceil + 2$. Recall that we would like to have the maximal length of $Q_1, Q_2, \ldots, Q_{k+1}$ bounded above by $\lceil k/2 \rceil + 1$. Besides, each path from s to d_i produced by Paths1 $(\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$ has length $|d_i|$ if $d_{i,\Phi(d_i)} = 1$, and $|d_i| + 2$ if $d_{i,\Phi(d_i)} = 0$, where $1 \leqslant i \leqslant k$ (refer to Lemma 1).

When $|d_{k+1}| \leqslant \lceil k/2 \rceil - 1$ (i.e., Cases 1 and 2), we have $|d_i| + 2 \leqslant (\lceil k/2 \rceil - 1) + 2 = \lceil k/2 \rceil + 1$ for all $1 \leqslant i \leqslant k$. Hence $Q_1, Q_2, \ldots, Q_k$ can be obtained by invoking Paths1 $(\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$. When $|d_{k+1}| \leqslant \lceil k/2 \rceil - 2$ (i.e., Case 1), the construction of Q_{k+1} depends on whether d_{k+1} is contained in these k paths or not. If d_{k+1} is contained internally in Q_l for some $1 \leqslant l \leqslant k$, then Q_{k+1} is constructed as the subpath of Q_l from s to d_{k+1} and Q_l needs to be reconstructed. Otherwise, Q_{k+1} is constructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_{k+1}}$ in a k-cube, and a link $(\overline{d_{k+1}}, d_{k+1})$.

When $|d_{k+1}| = \lceil k/2 \rceil - 1$ (i.e., Case 2), Q_{k+1} can be obtained all the same as Case 1 if k is even. If k is odd, then Q_{k+1} is constructed differently in order to maintain the upper bound (i.e., $\lceil k/2 \rceil + 1$) of the maximal length. We only show the difference as follows. If d_{k+1} is contained internally in Q_l for some $1 \leqslant l \leqslant k$, then Q_l is reconstructed, depending on whether d_{k+1} is the immediate predecessor of d_l in Q_l or not. Otherwise, Q_{k+1} is constructed as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube. If Q_{k+1} intersects with Q_h for some $1 \leqslant h \leqslant k$, then Q_{k+1} and Q_h need to be reconstructed.

On the other hand, when $|d_{k+1}| \geqslant \lceil k/2 \rceil$ (i.e., Cases 3–5), the paths produced by Paths1 $(\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$ may have lengths greater than $\lceil k/2 \rceil + 1$. Hence, destination nodes need to be changed. When $|d_{k+1}| = \lceil k/2 \rceil$ (i.e., Case 3), each d_i is replaced with $\overline{d_i}$ if $|d_i| = \lceil k/2 \rceil$ and $d_{i,\Phi(d_i)} = 0$, where $1 \leqslant i \leqslant k$. We have $|d_i| = \lceil k/2 \rceil$ and $d_{i,\Phi(d_i)} = 0$ for all $1 \leqslant i \leqslant r$. If Paths1$(\Phi, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$ was invoked to produce $Q_1, Q_2, \ldots, Q_k$, then each Q_i $(1 \leqslant i \leqslant r)$ would have length

$|d_i| + 2 = \lceil k/2 \rceil + 2$ according to Lemma 1. In order to minimize the maximal length, Paths1($\Omega, k, k, \{\overline{d}_1, \ldots, \overline{d}_r, d_{r+1}, \ldots, d_k\}, \{1, 2, \ldots, k\}$) is invoked to obtain $Q'_1, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$, instead. It is shown later that each Q'_i has length $|\overline{d}_i| = \lfloor k/2 \rfloor$. Then each Q_i can be obtained from Q'_i, depending on whether $\overline{d}_i \in \{\overline{d}_1, \ldots, \overline{d}_{i-1}, \overline{d}_{i+1}, \ldots, \overline{d}_r, d_{r+1}, \ldots, d_{k+1}\}$ or not. For either situation, the length of Q_i is greater than the length of Q'_i by one.

Q_{k+1} is constructed as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube. If Q_{k+1} intersects with Q_t for some $1 \leqslant t \leqslant r$ at a node $x \notin \{s, d_{k+1}\}$, then the reconstruction of Q_t further depends on whether $\overline{d}_t \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ or not. When $\overline{d}_t \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, Q_m needs to be reconstructed, where $d_m = \overline{d}_t$. If Q_{k+1} intersects with Q_l for some $r+1 \leqslant l \leqslant k$ at a node $(\neq s)$, then d_{k+1} is contained internally in Q_l, which is shown later. Q_{k+1} and Q_l need to be reconstructed. When k is odd and (new) Q_l intersects with Q_v for some $1 \leqslant v \leqslant r$ at a node $z(\neq s)$, Q_v and Q_w are reconstructed similar to Q_t and Q_m, where $d_w = \overline{d}_v$. Also Q_u needs to be reconstructed if $\overline{d}_v \in Q_u$ for some $1 \leqslant u \leqslant r$ and $u \neq v$.

When $|d_{k+1}| = \lceil k/2 \rceil + 1$ (i.e., Case 4), each d_i is replaced with $\overline{d}_i$ if $|d_i| = \lceil k/2 \rceil + 1$ and $d_{i,\Phi(d_i)} = 0$, where $1 \leqslant i \leqslant k$. Then d_l is replaced with $\overline{d}_l$ or p_l whenever $|d_l| = \lceil k/2 \rceil$, $d_l \in D$, and $d_{l,\Psi(d_l)} = 0$, where $r+1 \leqslant l \leqslant k$ and $D = \{\overline{d}_1, \ldots, \overline{d}_r, d_{r+1}, \ldots, d_k\}$ initially. We note that $d_{r+1}, d_{r+2}, \ldots, d_k$ have to be examined sequentially, because there may be multiple occurrences of d_l in $\{d_{r+1}, d_{r+2}, \ldots, d_k\}$. $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ are obtained by invoking Paths1($\Psi, k, k, \{\overline{d}_1, \ldots, \overline{d}_u, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_k\}, \{1, 2, \ldots, k\}$). Q_{k+1} can be obtained all the same as Case 3, but no reconstruction is needed.

For all $1 \leqslant i \leqslant r$, Q_i is constructed depending on three situations: (1) $\overline{d}_i \in \{d_{v+1}, d_{v+2}, \ldots, d_k\}$, (2) $\overline{d}_i \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ but $\overline{d}_i \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d}_r\}$, and (3) $\overline{d}_i \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\} \cup \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}\}$. For (1), Q_i is constructed as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f}_i)$ and $(\overline{f}_i, d_i)$. Before the construction of Q_i, Q'_i and Q_w need to be swapped provided $\overline{f}_i \in Q_{k+1}$, where $\overline{d}_i = d_w$. For (2), the construction of Q_i further depends on whether $\overline{f}_i \in Q_{k+1}$ or not. If $\overline{f}_i \in Q_{k+1}$, then Q_i can be obtained by augmenting Q'_i with a link $(\overline{d}_i, d_i)$. Otherwise, Q_i is constructed the same as (1). For (3), Q_i is constructed the same as (2) with $\overline{f}_i \in Q_{k+1}$. For all $r+1 \leqslant j \leqslant u$, Q_j can be obtained by augmenting Q'_j with a link $(\overline{d}_j, d_j)$. For all $u+1 \leqslant t \leqslant v$, Q_t can be obtained by augmenting Q'_t with two links $(p_t, \overline{p}_t)$ and $(\overline{p}_t, d_t)$.

When $|d_{k+1}| \geqslant \lceil k/2 \rceil + 2$ (i.e., Case 5), each d_i with $|d_i| \geqslant \lceil k/2 \rceil + 2$ (even if $d_{i,\Phi(d_i)} = 1$) is replaced with $\overline{d}_i$, where $1 \leqslant i \leqslant k$. Further each $\overline{d}_h$ with $|\overline{d}_h| = \lfloor k/2 \rfloor - 2$ is replaced with q_h conditionally (refer to statement (74*)), where $1 \leqslant h \leqslant u'$. Similarly, $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_{u'}$ have to be examined sequentially. After the replacement, each destination node $d'_i \in \{d_i, \overline{d}_i, q_i\}$ has $|d'_i| \leqslant \lceil k/2 \rceil + 1$. Hence, $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$ can be obtained by the construction method of Case 4. For all $1 \leqslant i \leqslant r'$, Q_i can be obtained by augmenting Q'_i with two links $(q_i, \overline{q}_i)$ and $(\overline{q}_i, d_i)$. For all $r'+1 \leqslant j \leqslant u'$, Q_j can be obtained also from Q'_j, which is similar to the construction of Q_i from Q'_i in Case 4.

As an illustrative example, suppose $k = 5$ and $(d_1, d_2, d_3, d_4, d_5, d_6) = (00111, 00101, 00111, 00110, 11100, 10011)$. We have $|d_{k+1}| = |d_6| = 3$ in Step 1. Since $|d_6| = 3 = \lceil 5/2 \rceil = \lceil k/2 \rceil$, Case 3 of Step 2 is executed. There is a minimal routing function $\Phi : \{d_1, d_2, d_3, d_4, d_5\} \rightarrow \{1, 2, 3, 4, 5\}$ defined as follows: $(\Phi(d_1), \Phi(d_2), \Phi(d_3), \Phi(d_4),$

$\Phi(d_5)) = (2, 3, 5, 4, 1)$. We have $d_{1,\Phi(d_1)} = 0$ and $d_{2,\Phi(d_2)} = d_{3,\Phi(d_3)} = d_{4,\Phi(d_4)} = d_{5,\Phi(d_5)} = 1$. The set $\{d_1, d_2, d_3, d_4, d_5\}$ is partitioned into $\{d_1\}$ and $\{d_2, d_3, d_4, d_5\}$, i.e., $r = 1$. The minimal routing function $\Omega : \{\overline{d}_1, d_2, d_3, d_4, d_5\} \to \{1, 2, 3, 4, 5\}$ is defined as follows: $(\Omega(\overline{d}_1), \Omega(d_2), \Omega(d_3), \Omega(d_4), \Omega(d_5)) = (\Phi(d_1), \Phi(d_2), \Phi(d_3), \Phi(d_4), \Phi(d_5)) = (2, 3, 5, 4, 1)$.

Then executing Paths1$(\Omega, 5, 5, \{\overline{d}_1, d_2, d_3, d_4, d_5\}, \{1, 2, 3, 4, 5\})$ can produce $Q'_1 = (00000, 01000, 11000)$, $Q_2 = (00000, 00100, 00101)$, $Q_3 = (00000, 00001, 00011, 00111)$, $Q_4 = (00000, 00010, 00110)$ and $Q_5 = (00000, 10000, 10100, 11100)$, which are disjoint paths from s to $\overline{d}_1, d_2, d_3, d_4$, and d_5, respectively. We note that Q'_1, Q_2, Q_3, Q_4, and Q_5 are all shortest, as a consequence of $\overline{d}_{1,\Omega(\overline{d}_1)} = d_{2,\Omega(d_2)} = d_{3,\Omega(d_3)} = d_{4,\Omega(d_4)} = d_{5,\Omega(d_5)} = 1$. Since $\overline{d}_1 \notin \{d_2, d_3, \ldots, d_6\}$, $Q_1 = (00000, 01000, 11000, 00111)$ can be obtained by augmenting Q'_1 with link $(\overline{d_1}, d_1) = (11000, 00111)$.

$Q_6 = (00000, 11111, 10111, 10011)$ can be obtained by combining $(s, \bar{s}) = (00000, 11111)$ and a shortest path, i.e., $(11111, 10111, 10011)$, from $\bar{s}$ to d_6 in a 5-cube. Since Q_6 is disjoint with Q_1 and d_6 is not contained internally in Q_2, Q_3, Q_4 and Q_5, no further reconstruction is needed. $Q_1, Q_2, \ldots, Q_6$ are all disjoint whose lengths are not greater than three.

3.2. *Procedure Paths4*$(s, d_1, d_2, \ldots, d_{k+1})$

We also use $Q_1, Q_2, \ldots, Q_{k+1}$ to denote the $k+1$ paths produced by Paths4$(s, d_1, d_2, \ldots, d_{k+1})$, where Q_i $(1 \leqslant i \leqslant k+1)$ is the path from s to d_i. Recall that Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ assumes that two or more of $d_1, d_2, \ldots, d_{k+1}$ are 1^k. The following is a formal description of Paths4$(s, d_1, d_2, \ldots, d_{k+1})$.

Procedure Paths4$(s, d_1, d_2, \ldots, d_{k+1})$.

/* Without loss of generality, assume $d_{k-r+1} = d_{k-r+2} = \cdots = d_{k+1} = 1^k$ and $d_i \neq 1^k$ for all $1 \leqslant i \leqslant k - r$, where $1 \leqslant r \leqslant k$. */

Step 1: Determine r (distinct) adjacent nodes, denoted by $c_{k-r+1}, c_{k-r+2}, \ldots, c_k$, of s in a k-cube so that $c_j \notin \{d_1, d_2, \ldots, d_{k-r}\}$ and $\overline{c_j} \notin \{d_1, d_2, \ldots, d_{k-r}\}$ for all $k - r + 1 \leqslant j \leqslant k$.

Step 2: Construct $Q_1, \ldots, Q_{k-r}, Q'_{k-r+1}, \ldots, Q'_k, Q_{k+1}$ in a k-fcube by invoking Paths3 $(s, d_1, \ldots, d_{k-r}, c_{k-r+1}, \ldots, c_k, d_{k+1})$.

/* Assume that Q'_j ends at c_j for all $k - r + 1 \leqslant j \leqslant k$. */

For $j = k - r + 1,\ k - r + 2, \ldots, k$,

Construct Q_j as the combination of three links (s, c_j), $(c_j, \overline{c_j})$, and $(\overline{c_j}, \bar{s})$.

The execution of Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ first replaces the destination nodes $d_{k-r+1}, d_{k-r+2}, \ldots, d_k$ with $c_{k-r+1}, c_{k-r+2}, \ldots, c_k$ so that Paths3$(s, d_1, \ldots, d_{k-r}, c_{k-r+1}, \ldots, c_k, d_{k+1})$ can be invoked to produce $Q_1, \ldots, Q_{k-r}, Q'_{k-r+1}, \ldots, Q'_k, Q_{k+1}$. Then, for all $k - r + 1 \leqslant j \leqslant k$, each Q'_j is replaced with Q_j, which is composed of links (s, c_j), $(c_j, \overline{c_j})$, and $(\overline{c_j}, \bar{s})$.

3.3. *Main result*

It will be shown in Section 4 that $Q_1, Q_2, \ldots, Q_{k+1}$ produced by Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ and Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ are disjoint with maximal length not greater than

$\lceil k/2 \rceil + 1$, where $k \geqslant 4$. This imposes an upper bound of $\lceil k/2 \rceil + 1$ on the strong Rabin number of a k-fcube. On the other hand, the Rabin number of a k-fcube is known to be $\lceil k/2 \rceil + 1$ (see [11,12]), which provides a lower bound of $\lceil k/2 \rceil + 1$. Consequently, the strong Rabin number of a k-fcube is equal to $\lceil k/2 \rceil + 1$. It is not difficult to check that the strong Rabin numbers of a folded 2-cube and a folded 3-cube are equal to 2 and 3, respectively. Hence, we have the following theorem, which is the main result of this paper.

Theorem 1. *The strong Rabin number of a k-dimensional folded hypercube is equal to* $\lceil k/2 \rceil + 1$, *where* $\lceil k/2 \rceil$ *is the diameter*.

4. The proof of Theorem 1

It suffices to show that $Q_1, Q_2, \ldots, Q_{k+1}$ produced by Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ and Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ are disjoint with maximal length not greater than $\lceil k/2 \rceil + 1$. Section 4.1 assumes that at most one of $d_1, d_2, \ldots, d_{k+1}$ is 1^k (i.e., Paths3$(s, d_1, d_2, \ldots, d_{k+1})$). Section 4.2 assumes that two or more of $d_1, d_2, \ldots, d_{k+1}$ are 1^k (i.e., Paths4$(s, d_1, d_2, \ldots, d_{k+1})$).

4.1. When at most one of $d_1, d_2, \ldots, d_{k+1}$ is 1^k

In this section, we show that $Q_1, Q_2, \ldots, Q_{k+1}$ produced by Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ are disjoint with maximal length not greater than $\lceil k/2 \rceil + 1$. The following lemma was shown in [11].

Lemma 2 (*Lai [11]*). *Suppose that* $D = \{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ *results after the statement* (74*), *where* $0 \leqslant r' \leqslant u' \leqslant k$. *If* $\Gamma : D \to \{1, 2, \ldots, k\}$ *is minimal, then* $q_{i,\Gamma(q_i)} = 1$ *for all* $1 \leqslant i \leqslant r'$.

The execution of Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ contains two steps, i.e., Steps 1 and 2. Step 1 is easy and hence no further explanation is needed. Step 2 constructs $Q_1, Q_2, \ldots, Q_{k+1}$ according to five cases, i.e., Cases 1–5. We have to show that all the five cases produce disjoint paths $Q_1, Q_2, \ldots, Q_{k+1}$ whose lengths are not greater than $\lceil k/2 \rceil + 1$. Recall that Paths3$(s, d_1, d_2, \ldots, d_{k+1})$ is an extension of Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. In order to show the correctness of Paths3$(s, d_1, d_2, \ldots, d_{k+1})$, we need to extend the proof for Paths2$(s, d_1, d_2, \ldots, d_{k+1})$ which appeared in [11,12].

The proof for Case 1 can be obtained by replacing all the occurrences of "$d_{k+1} \in Q_l$" and "$d_{k+1} \notin Q_i$" in Section 4.1 of [12] (or Section 4.1.2.1 of [11]) with "$d_{k+1} \hat{\in} Q_l$" and "$d_{k+1} \hat{\notin} Q_i$", respectively. Similarly, the proof for Case 2 can be obtained by replacing all the occurrences of "$d_{k+1} \in Q_l$", "$d_{k+1} \in (\text{old})Q_l$", "$d_{k+1} \notin Q_i$", "$d_{k+1} \notin (\text{old})Q_h$" and "$(\neq s)$" in Section 4.2 of [12] (or Section 4.1.2.2 of [11]) with "$d_{k+1} \hat{\in} Q_l$", "$d_{k+1} \hat{\in} (\text{old})Q_l$", "$d_{k+1} \hat{\notin} Q_i$", "$d_{k+1} \hat{\notin} (\text{old})Q_h$" and "$(\notin \{s, d_{k+1}\})$", respectively. We note that the path Q_{k+1} obtained when $d_{k+1} \hat{\notin} Q_i$ for all $1 \leqslant i \leqslant k$ is disjoint with Q_h if their intersection belongs to $\{s, d_{k+1}\}$ (refer to the statement (26*)), where $1 \leqslant h \leqslant k$. Hence no reconstruction is needed.

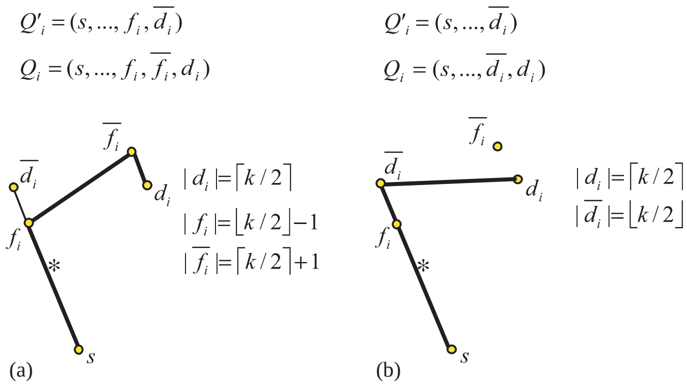

Fig. 2. The construction of Q_i (indicated by darkened edges) from Q'_i, where $1 \leqslant i \leqslant r$ (* denotes a shortest path). (a) When $\overline{d_i} \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_{k+1}\}$. (b) When $\overline{d_i} \notin \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_{k+1}\}$.

In Sections 4.1.1, 4.1.2 and 4.1.3, we outline the proofs for Cases 3, 4 and 5, respectively. Some omitted details can be found in Section 4.1.2 of [11] or Section 4 of [12].

4.1.1. When $|d_{k+1}| = \lceil k/2 \rceil$

First, $Q'_1, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$ were produced by Paths1 $(\Omega, k, k, \{\overline{d_1}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}, \{1, 2, \ldots, k\})$, where Q'_i ends at $\overline{d_i}$ for all $1 \leqslant i \leqslant r$. It was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]) that they are disjoint. Besides, for all $1 \leqslant i \leqslant r$, each Q'_i is shortest with length $|\overline{d_i}| = \lfloor k/2 \rfloor$. For all $r + 1 \leqslant j \leqslant k$, each Q_j is shortest with length $|d_j| = \lceil k/2 \rceil$ if $|d_j| = \lceil k/2 \rceil$, and has length at most $\lceil k/2 \rceil + 1$ if $|d_j| \leqslant \lceil k/2 \rceil - 1$. We note $|d_j| \leqslant |d_{k+1}| = \lceil k/2 \rceil$.

Then, for all $1 \leqslant i \leqslant r$, each Q_i was obtained from Q'_i, depending on whether $\overline{d_i} \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_{k+1}\}$ or not. Refer to Fig. 2. The construction of Q_i is the same as the construction of Q_i in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Clearly, each Q_i has length $\lfloor k/2 \rfloor + 1$ and $Q_1, Q_2, \ldots, Q_k$ are disjoint provided $Q_1, Q_2, \ldots, Q_r$ are disjoint and each Q_i is disjoint with $Q_{r+1}, Q_{r+2}, \ldots, Q_k$. We have $Q_1, Q_2, \ldots, Q_r$ disjoint if the following four conditions hold: (1) $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}$ are all distinct, (2) $\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}\} \cap \{d_1, d_2, \ldots, d_r\}$ is empty, (3) each of $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, d_1, d_2, \ldots, d_r$ does not belong to $Q'_1, Q'_2, \ldots, Q'_r$, and (4) $\overline{d_i}$ belongs to at most one of $Q_1, Q_2, \ldots, Q_r$. Conditions (1), (2) and (3) were shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]). Condition (4) can be assured by condition (3) and the construction of Q_i. It was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]) that each Q_i is disjoint with $Q_{r+1}, Q_{r+2}, \ldots, Q_k$.

Q_{k+1} was constructed all the same as in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. $Q_1, Q_2, \ldots, Q_r$ were then examined and reconstruction was needed if they are not disjoint with Q_{k+1}. Without loss of generality, we assume that Q_{k+1} intersects with Q_t at a node $x \notin \{s, d_{k+1}\}$, where $1 \leqslant t \leqslant r$. We have $x = \overline{f_t}$ and $|x| = \lceil k/2 \rceil + 1$, which was shown in Section 4.1.2.3 of [11]. When $x = d_{k+1}$, we have $d_{k+1} = d_t$ for the following reason. Since Q'_t is shortest,

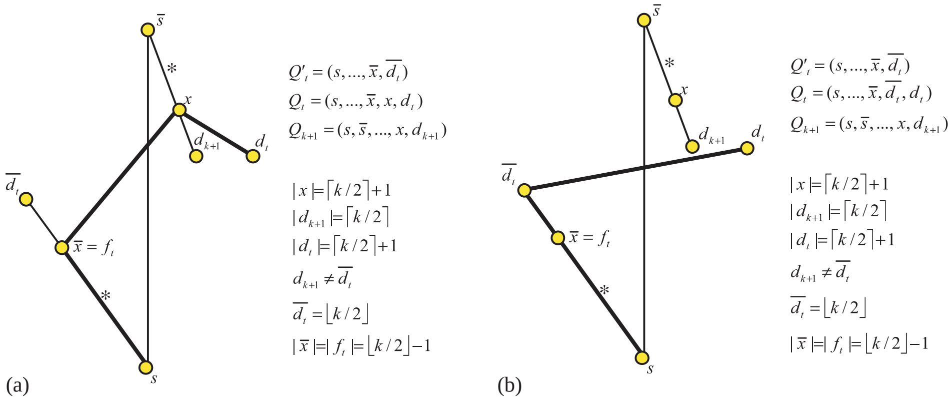

Fig. 3. The reconstruction of Q_t (indicated by darkened edges) when $\overline{d_t} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$. (a) Before reconstruction. (b) After reconstruction.

$|d_{k+1}| = \lceil k/2 \rceil > \lfloor k/2 \rfloor - 1 = |f_t|$ and $|d_{k+1}| \neq \lceil k/2 \rceil + 1 = |\overline{f_t}|$, the construction of Q_t assures $d_{k+1} \hat{\notin} Q_t$, which implies $d_{k+1} = d_t$.

The reconstruction depends on whether $\overline{d_t} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ or not. When $\overline{d_t} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, the reconstruction is all the same as in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. It was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]) that after the reconstruction, $Q_1, Q_2, \ldots, Q_k$ are disjoint with maximal length not greater than $\lceil k/2 \rceil + 1$. Besides, Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_r$. When $\overline{d_t} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, Q_t needs to be reconstructed. Refer to Fig. 3. Since $\bar{x} = f_t$, (new) Q_t is actually the combination of Q'_t and a link $(\overline{d_t}, d_t)$ whose length is $\lfloor k/2 \rfloor + 1$. It was shown in Section 4.1.2.3 of [11] that $\overline{d_t}$ is not contained in $Q_1, \ldots, Q_{t-1}, Q_{t+1}, \ldots, Q_k$. This assures that (new) Q_t is disjoint with $Q_1, \ldots, Q_{t-1}, Q_{t+1}, \ldots, Q_k$.

$Q_1, \ldots, Q_{t-1}$, (new) $Q_t, Q_{t+1}, \ldots, Q_r$ are disjoint with Q_{k+1}, as explained below. It was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]) that at most one of $Q_1, \ldots, Q_{t-1}$, (old) $Q_t, Q_{t+1}, \ldots, Q_r$ is not disjoint with Q_{k+1}. It follows that $Q_1, \ldots, Q_{t-1}, Q_{t+1}, \ldots, Q_r$ are disjoint with Q_{k+1}. Besides, (new) Q_t is disjoint with Q_{k+1}, as shown in Fig. 3(b), where $|x| = \lceil k/2 \rceil + 1$, $|\overline{d_t}| = \lfloor k/2 \rfloor$, $|d_t| = \lceil k/2 \rceil$, and $\overline{d_t} \neq d_{k+1}$ (refer to Section 4.1.2.3 of [11]). The subpath of Q_{k+1} from $\bar{s}$ to x and the subpath of (new) Q_t from s to $\bar{x}$ are shortest in a k-cube.

To sum up, after $Q_1, Q_2, \ldots, Q_r$ were examined and necessary reconstruction was made, $Q_1, Q_2, \ldots, Q_k$ remain disjoint and Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_r$ (Q_{k+1} is not necessarily disjoint with $Q_{r+1}, Q_{r+2}, \ldots, Q_k$).

Then $Q_{r+1}, Q_{r+2}, \ldots, Q_k$ were examined and reconstruction was needed if they are not disjoint with Q_{k+1}. Suppose that α is an arbitrary node in $Q_{r+1}, Q_{r+2}, \ldots, Q_k$ and Q_{k+1} intersects with Q_l at a node $x'(\neq s)$, where $r + 1 \leqslant l \leqslant k$. We have $|\alpha| \leqslant \lceil k/2 \rceil$, which was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]). Since $|d_{k+1}| = \lceil k/2 \rceil \geqslant |\alpha|$ and the subpath of Q_{k+1} from $\bar{s}$ to d_{k+1} is shortest in a k-cube, we have $x' = d_{k+1}$. If $d_{k+1} = d_l$, then Q_{k+1} is disjoint with Q_l.

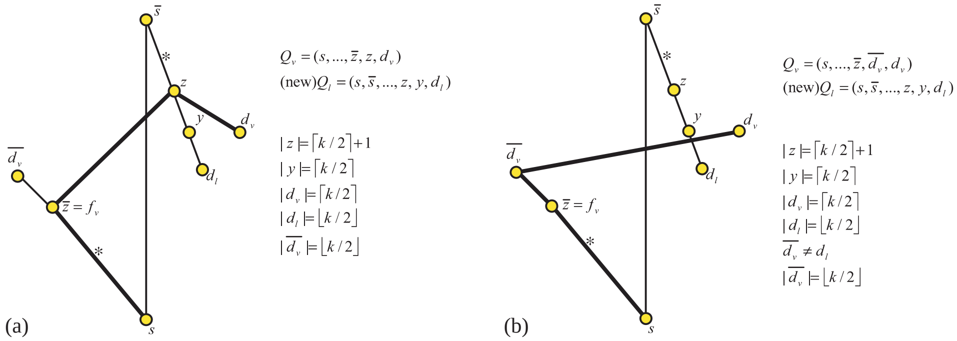

Fig. 4. The reconstruction of Q_v (indicated by darkened edges) when $\overline{d}_v \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$. (a) Before reconstruction. (b) After reconstruction.

If $d_{k+1} \neq d_l$ (or $d_{k+1} \hat{\in} Q_l$), then Q_{k+1} and Q_l were reconstructed all the same as in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. Clearly, (new) Q_{k+1} (a subpath of (old) Q_l) is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$. It was shown in Section 4.3 of [12] (or Section 4.1.2.3 of [11]) that (new) Q_l is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1} if k is even, and $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1} if k is odd. When k is odd, (new) Q_l is not necessarily disjoint with $Q_1, Q_2, \ldots, Q_r$ and further reconstruction is needed if they are not disjoint.

Suppose that (new) Q_l intersects with Q_v at a node z $(\neq s)$, where $1 \leqslant v \leqslant r$. We have $|d_l| = \lceil k/2 \rceil - 1$, $z = \overline{f_v}$, $|z| = \lceil k/2 \rceil + 1$ and $\overline{d_v} \in \{\overline{d_1}, \ldots, \overline{d_{v-1}}, \overline{d_{v+1}}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}$, which was shown in Section 4.1.2.3 of [11]. The reconstruction depends on whether $\overline{d_v} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ or not. If $\overline{d_v} \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then the reconstruction is the same as that in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$ and the resulting $Q_1, Q_2, \ldots, Q_{k+1}$ are disjoint. If $\overline{d_v} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$, then Q_v was reconstructed as the combination of the subpath of (old) Q_v from s to $\bar{z}$, a shortest path from $\bar{z}$ to $\overline{d_v}$, and a link $(\overline{d_v}, d_v)$. Refer to Fig. 4. Since $z = \overline{f_v}$ (or $\bar{z} = f_v$), (new) Q_v has length $(\lfloor k/2 \rfloor - 1) + 2 = \lfloor k/2 \rfloor + 1$.

(New) Q_l is disjoint with $Q_1, Q_2, \ldots,$ (new) $Q_v, \ldots, Q_r$, as explained below. We first show that (new) Q_l is disjoint with $Q_1, \ldots, Q_{v-1}, Q_{v-1}, \ldots, Q_r$. Suppose conversely that (new) Q_l intersects with $Q_{v'}$ at a node $z' (\neq s)$, where $1 \leqslant v' \leqslant r$ and $v' \neq v$. We have $|z'| = \lceil k/2 \rceil + 1$, which can be shown similar to $|z| = \lceil k/2 \rceil + 1$ above. Since $|z'| = |z| > \lceil k/2 \rceil = |d_l| + 1 = |y|$ and the subpath of (new) Q_l from $\bar{s}$ to y is shortest in a k-cube, we have $z' = z$, which means that $Q_{v'}$ and (old) Q_v are not disjoint. This is a contradiction. (New) Q_l is disjoint with (new) Q_v, as explained by Fig. 4(b), where $|y| = |d_v|$, $y \neq d_v$ ($y \notin$ (old) Q_v), $|y| > |\overline{d_v}| > |\bar{z}|$, $|\overline{d_v}| = |d_l|$, and $\overline{d_v} \neq d_l$ (refer to Section 4.3 of [12] or Section 4.1.2.3 of [11]).

(New) Q_v is disjoint with $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1}, as explained below. Since $\overline{f_v} = z \in$ (old) Q_v, we have (old) Q_v being the combination of the subpath of Q'_v from s to f_v and two links $(f_v, \overline{f_v})$ and $(\overline{f_v}, d_v)$ and (new) Q_v being the combination of Q'_v and a link $(\overline{d_v}, d_v)$. Recall that (old) Q_v is disjoint with $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1}. (New) Q_v is disjoint with $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1},

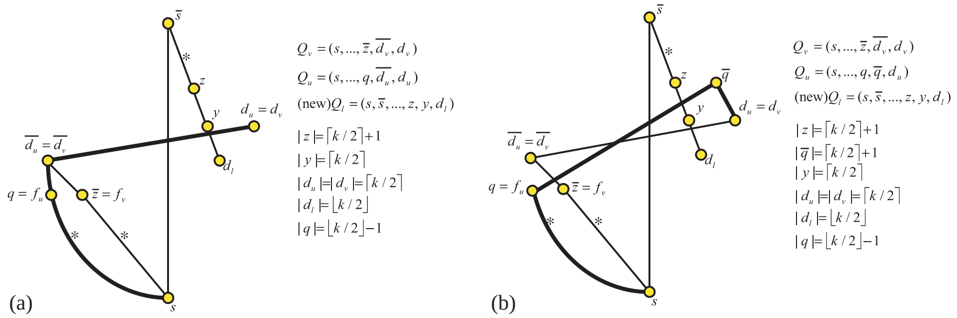

Fig. 5. The reconstruction of Q_u (indicated by darkened edges) when $\overline{d}_v \in Q_u$ for some $1 \leqslant u \leqslant r$ and $u \neq v$. (a) Before reconstruction. (b) After reconstruction.

provided $\overline{d_v}$ is not contained in $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1}. The latter is true because Q'_v is disjoint with $Q_{r+1}, \ldots, Q_{l-1}$, (old) $Q_l, Q_{l+1}, \ldots, Q_k$ and $\overline{d_v} \notin \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ ((new) Q_{k+1} is a subpath of (old) Q_l). (New) Q_v is not necessarily disjoint with $Q_1, \ldots, Q_{v-1}, Q_{v+1}, \ldots, Q_r$.

Suppose that (new) Q_v is not disjoint with Q_u, where $1 \leqslant u \leqslant r$ and $u \neq v$. Recall that (old) Q_v is disjoint with Q_u. We have $\overline{d_v}$ being the intersection of (new) Q_v and Q_u because $\overline{d_v}$ is the unique node of (new) Q_v that is not contained in (old) Q_v. Q_u was reconstructed as the combination of the subpath of (old) Q_u from s to q, a link $(q, \bar{q})$ and a shortest path from $\bar{q}$ to d_u, where q is the immediate predecessor of $\overline{d_v}$ in Q_u. Refer to Fig. 5. (New) Q_u has length $(\lfloor k/2 \rfloor - 1) + 2 = \lfloor k/2 \rfloor + 1$ because $q = f_u$ (refer to Section 4.1.2.3 of [11]).

Recall that (new) Q_l is disjoint with $Q_1, \ldots, Q_{u-1}, Q_{u+1}, \ldots,$ (new) $Q_v, \ldots, Q_r$, where $u < v$ is assumed. (New) Q_l is disjoint with (new) Q_u as explained by Fig. 5(b), where $|y| = |d_u|$, $y \neq d_u$ ($y \notin$ (old) Q_u), $|y| > |q|$, $|d_l| > |q|$, $|\bar{q}| = |z| > |y| > |d_l|$, $\bar{q} \neq z$, and the subpath of (new) Q_l from $\bar{s}$ to y is shortest in a k-cube. It was shown in Section 4.1.2.3 of [11] that (new) Q_u is disjoint with $Q_1, \ldots, Q_{u-1}, Q_{u+1}, \ldots,$ (new) $Q_v, \ldots, Q_r, Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1}.

4.1.2. When $|d_{k+1}| = \lceil k/2 \rceil + 1$

First, $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ were produced by Paths1($\Psi, k, k, D, \{1, 2, \ldots, k\}$), where Q'_j ends at $\overline{d_j}$ for all $1 \leqslant j \leqslant u$ and Q'_t ends at p_t for all $u+1 \leqslant t \leqslant v$. It was shown in Section 4.1.2.4 of [11] that $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ are disjoint. Besides, Q'_i is shortest with length $|\overline{d_i}| = \lfloor k/2 \rfloor - 1$ for all $1 \leqslant i \leqslant r$, Q'_j is shortest with length $|\overline{d_j}| = \lfloor k/2 \rfloor$ for all $r+1 \leqslant j \leqslant u$, Q'_t is shortest with length $|p_t| = \lfloor k/2 \rfloor - 1$ for all $u+1 \leqslant i \leqslant v$, and Q_c has length not greater than $\lceil k/2 \rceil + 1$ for all $v+1 \leqslant c \leqslant k$.

Q_{k+1} was constructed all the same as in Paths2($d_1, d_2, \ldots, d_{k+1}$) whose length is $\lfloor k/2 \rfloor$. That Q_{k+1} is disjoint with $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ was shown in Section 4.4 of [12] (or Section 4.1.2.4 of [11]).

Then, for all $1 \leqslant i \leqslant r$, each Q_i was obtained from Q'_i according to three situations: (1) $\overline{d_i} \in \{d_{v+1}, d_{v+2}, \ldots, d_k\}$, (2) $\overline{d_i} \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ but $\overline{d_i} \in \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}\}$, and (3) $\overline{d_i} \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\} \cup \{\overline{d_1}, \ldots, \overline{d_{i-1}}, \overline{d_{i+1}}, \ldots, \overline{d_r}\}$. The constructions for (1) and (3) are the same as those in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. It was shown in Section 4.4 of [12] (or Section 4.1.2.4 of [11]) that each Q_i has length $\lfloor k/2 \rfloor$ and is disjoint with $Q'_{r+1}, \ldots, Q'_v, Q_{v+1}, \ldots, Q_{k+1}$. The construction for (2) is the same as that for (3) if $\overline{f_i} \in Q_{k+1}$, and is the same as that for (1) if $\overline{f_i} \notin Q_{k+1}$.

Clearly, $Q_1, Q_2, \ldots, Q_r$ are disjoint provided the following four conditions hold: $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}$ are all distinct, $\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}\} \cap \{d_1, d_2, \ldots, d_r\}$ is empty, each of $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, d_1, d_2, \ldots, d_r$ is not contained in $Q'_1, Q'_2, \ldots, Q'_r$, and each $\overline{d_i}$ belongs to at most one of $Q_1, Q_2, \ldots, Q_r$. The first three conditions hold because $f_1, f_2, \ldots, f_r$ are all distinct and for all $1 \leqslant c \leqslant r$, $|f_c| = \lfloor k/2 \rfloor - 2$ (or $|\overline{f_c}| = \lceil k/2 \rceil + 2$), $|d_c| = \lceil k/2 \rceil + 1$, and each Q'_c is shortest from s to $\overline{d_c}$ in a k-cube. That each $\overline{d_i}$ belongs to at most one of $Q_1, Q_2, \ldots, Q_r$ was shown in Section 4.1.2.4 of [11].

$Q_{r+1}, Q_{r+2}, \ldots, Q_v$ were constructed all the same as in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. It was shown in Section 4.1.2.4 of [11] that they are disjoint with length $\lfloor k/2 \rfloor + 1$. Besides, each of them is disjoint with $Q_1, \ldots, Q_r, Q_{v+1}, \ldots, Q_{k+1}$.

4.1.3. When $|d_{k+1}| \geqslant \lfloor k/2 \rfloor + 2$

It is not difficult to check that the $k+1$ disjoint paths produced by Case 4 (i.e., Section 4.1.2) remain disjoint and have maximal length not greater than $\lceil k/2 \rceil + 1$, even if $|d_i| \leqslant \lceil k/2 \rceil + 1$ for all $1 \leqslant i \leqslant k$ and $|d_{k+1}| \geqslant \lceil k/2 \rceil + 1$. Hence, $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$ can be obtained by the construction method of Case 4 with input Ω and $\{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_{k+1}\}$, where Ω is minimal, $|q_i| = \lfloor k/2 \rfloor - 1$ for all $1 \leqslant i \leqslant r'$, $|\overline{d_j}| \leqslant \lfloor k/2 \rfloor - 2$ for all $r'+1 \leqslant j \leqslant u'$, $|d_t| \leqslant \lceil k/2 \rceil + 1$ for all $u'+1 \leqslant t \leqslant k$, and $|d_{k+1}| \geqslant \lceil k/2 \rceil + 2$. By Lemma 2 we have $q_{i,\Omega(q_i)} = 1$ for all $1 \leqslant i \leqslant r'$. By Lemma 1, each Q'_i is shortest with length $|q_i| = \lfloor k/2 \rfloor - 1$, and each $Q'_j (r'+1 \leqslant j \leqslant u')$ has length not greater than $|\overline{d_j}| + 2 \leqslant (\lfloor k/2 \rfloor - 2) + 2 = \lfloor k/2 \rfloor$. There is no complement link contained in $Q'_1, Q'_2, \ldots, Q'_{u'}$.

Then, for all $1 \leqslant i \leqslant r'$, each Q_i was constructed as the combination of Q'_i and two links $(q_i, \overline{q_i})$ and $(\overline{q_i}, d_i)$ whose length is equal to $(\lfloor k/2 \rfloor - 1) + 2 = \lfloor k/2 \rfloor + 1$. It was shown in Section 4.1.2.5 of [11] that $Q_1, \ldots, Q'_{r'}, Q'_{r'+1}, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$ are disjoint.

For all $r'+1 \leqslant j \leqslant u'$, each Q_j was obtained from Q'_j according to three situations: (1) $\overline{d_j} \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$, (2) $\overline{d_j} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ but $\overline{d_j} \in \{\overline{d_{r'+1}}, \ldots, \overline{d_{j-1}}, \overline{d_{j+1}}, \ldots, \overline{d_{u'}}\}$, and (3) $\overline{d_j} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\} \cup \{\overline{d_{r'+1}}, \ldots, \overline{d_{j-1}}, \overline{d_{j+1}}, \ldots, \overline{d_{u'}}\}$. The constructions for (1) and (3) are the same as those in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$. The construction for (2) is the same as that for (3) if $\overline{g_j} \in Q_{k+1}$, and is the same as that for (1) if $\overline{g_j} \notin Q_{k+1}$. Each Q_j has length not greater than $\lfloor k/2 \rfloor + 1$. It was shown in Section 4.1.2.5 of [11] that each Q_j is disjoint with $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$.

Clearly, $Q_{r'+1}, Q_{r'+2}, \ldots, Q_{u'}$ are disjoint provided the following four conditions hold: $\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{u'}}$ are all distinct, $\{\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{u'}}\} \cap \{d_{r'+1}, d_{r'+2}, \ldots, d_{u'}\}$ is empty, each of $\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{u'}}, d_{r'+1}, d_{r'+2}, \ldots, d_{u'}$ is not contained in $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$, and $\overline{d_j}$ belongs to at most one of $Q_{r'+1}, Q_{r'+2}, \ldots, Q_{u'}$ for all $r'+1 \leqslant j \leqslant u'$. The first two conditions hold because $g_{r'+1}, g_{r'+2}, \ldots, g_{u'}$ are all distinct and $\{g_{r'+1}, g_{r'+2}, \ldots, g_{u'}\} \cap \{\overline{d_{r'+1}}, \overline{d_{r'+2}}, \ldots, \overline{d_{u'}}\}$ is empty. That the other two conditions hold was shown in Section 4.1.2.5 of [11].

4.2. When two or more of $d_1, d_2, \ldots, d_{k+1}$ are 1^k

In this section, we show that $Q_1, Q_2, \ldots, Q_{k+1}$ produced by Paths4$(s, d_1, d_2, \ldots, d_{k+1})$ are disjoint with maximal length not greater than $\lceil k/2 \rceil + 1$. Since there are k adjacent nodes of s in a k-cube, nodes $c_{k-r+1}, c_{k-r+2}, \ldots, c_k$ surely exist. Clearly, $Q_{k-r+1}, Q_{k-r+2}, \ldots, Q_k$ are disjoint with length three. Since one (i.e., d_{k+1}) of $d_1, d_2, \ldots, d_{k-r}, c_{k-r+1}, \ldots, c_k, d_{k+1}$ is 1^k, the proof of Section 4.1 assures that $Q_1, Q_2, \ldots, Q_{k-r}, Q_{k+1}$ are disjoint with lengths not greater than $\lceil k/2 \rceil + 1$. We show below that each Q_j is disjoint with $Q_1, Q_2, \ldots, Q_{k-r}, Q_{k+1}$, where $k - r + 1 \leqslant j \leqslant k$.

Since Q'_j is disjoint with $Q_1, Q_2, \ldots, Q_{k-r}, Q_{k+1}$, it suffices to show that $\overline{c_j}$ is not contained in $Q_1, Q_2, \ldots, Q_{k-r}, Q_{k+1}$. Since $|d_{k+1}| = k \geqslant \lceil k/2 \rceil + 2$, Q_{k+1} was constructed as a link $(s, \bar{s})$ by Paths3$(s, d_1, \ldots, d_{k-r}, c_{k-r+1}, \ldots, c_k, d_{k+1})$. Hence $\overline{c_j} \notin Q_{k+1}$. In the following, we show $\overline{c_j} \notin Q_i$ for all $1 \leqslant i \leqslant k - r$.

Suppose conversely $\overline{c_j} \in Q_i$. According to the execution of Paths3$(s, d_1, \ldots, d_{k-r}, c_{k-r+1}, \ldots, c_k, d_{k+1})$, each Q_i contains at most one complement link. Moreover, the complement link, if it exists, connects w_i and $\overline{w_i}$, where w_i is the immediate predecessor of d_i in Q_i. We have $\overline{c_j} \neq d_i$. If Q_i contains no complement link, then Q_i has length at least $|\overline{c_j}| + 1 = (k-1) + 1 = k > \lceil k/2 \rceil + 1$, which contradicts to the proof in Section 4.1. If Q_i contains one complement link, then $\overline{c_j} = w_i$, for otherwise Q_i has length at least $|\overline{c_j}| + 2 = k + 1 > \lceil k/2 \rceil + 1$, which again contradicts to the proof in Section 4.1. Hence the complement link is $(c_j, \overline{c_j})$. It follows that c_j belongs to Q_i. This is a contradiction because $c_j \neq d_i$ and Q'_j is disjoint with Q_i.

5. Concluding remarks

In this paper, we showed that the strong Rabin number of a k-dimensional folded hypercube is equal to $\lceil k/2 \rceil + 1$, where $\lceil k/2 \rceil$ is the diameter. Moreover, each disjoint path from s to d_i has length not greater than the distance from s to d_i plus two. In [13], Liaw and Chang showed that there exist $\lceil k/2 \rceil + 1$ distinct nodes $s, d_1, d_2, \ldots, d_{\lceil k/2 \rceil}$ in a k-dimensional folded hypercube so that any $\lceil k/2 \rceil$ disjoint paths from s to $d_1, d_2, \ldots, d_{\lceil k/2 \rceil}$, respectively, have maximal length greater than or equal to $\lceil k/2 \rceil + 1$. As a consequence of Theorem 1, Liaw and Chang's lower bound is tight.

In [13], the *w-Rabin number* of a network W was defined to be the minimum l so that for any $w + 1$ distinct nodes $s, d_1, d_2, \ldots, d_w$ of W, there exist w disjoint paths of lengths at most l from s to $d_1, d_2, \ldots, d_w$, respectively, where $1 \leqslant w \leqslant k$ and k is the connectivity of W. When $w = k$, the w-Rabin number of W is equal to the Rabin number of W. The *strong w-Rabin number* of W was also defined in [13], where $d_1, d_2, \ldots, d_w$ are not necessarily distinct. The w-Rabin numbers and strong w-Rabin numbers of some networks such as the hypercube, the d-ary hypercube, the generalized hypercube, the WK-recursive network, and a subclass of circulant networks were computed in [13]. The w-Rabin number and strong w-Rabin number of the folded hypercube are unknown.

Previously, strong Rabin numbers were computed only for recursive networks (those with recursive structures), e.g., the hypercube [7,13], the generalized hypercube [13], and the circulant network [13]. In this paper, we showed that routing functions can be used to compute the strong Rabin number of a nonrecursive network such as the folded hypercube. Routing functions have been shown effective in deriving disjoint paths in the hypercube and folded hypercube (see [11,12]). It appears that routing functions can be used to construct disjoint paths for other networks, especially, hypercubic networks. For example, refer to [7], where an SDR (System of Distinct Representatives) was used to construct disjoint paths in a k-ary hypercube. Actually, the SDR corresponds to a routing function (see [11]).

Appendix: Statements in Paths2$(s, d_1, d_2, \ldots, d_{k+1})$ that are different from those in Paths3$(s, d_1, d_2, \ldots, d_{k+1})$

(6) If $d_{k+1} \in Q_l$ for some $1 \leqslant l \leqslant k$, then {
(9) If $d_{k+1} \notin Q_i$ for all $1 \leqslant i \leqslant k$, then
(17) If $d_{k+1} \in Q_l$ for some $1 \leqslant l \leqslant k$, then {
(24) If $d_{k+1} \notin Q_i$ for all $1 \leqslant i \leqslant k$, then {
(26) If Q_{k+1} intersects with Q_h for some $1 \leqslant h \leqslant k$ at a node $y (\neq s)$, then {
(35) If $\overline{d_i} = d_h$ for some $r+1 \leqslant h \leqslant k+1$, then
(37) If $\overline{d_i} \neq d_j$ for all $r+1 \leqslant j \leqslant k+1$, then
(40) If Q_{k+1} intersects with Q_t for some $1 \leqslant t \leqslant r$ at a node $x\ (\neq s)$, then {
(41) /* Suppose that $\overline{d_t} = d_m$ for some $r+1 \leqslant m \leqslant k$ and g is the immediate predecessor of d_m in Q_m. */
(44) If $d_{k+1} \in Q_l$ for some $r+1 \leqslant l \leqslant k$, then {
(52) /* Suppose that $\overline{d_v} = d_w$ for some $r+1 \leqslant w \leqslant k$ and $w \neq l$, and p is the immediate predecessor of d_w in Q_w. */
(56) /* Suppose that $\Phi : \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_r\}$, $\{d_{r+1}, d_{r+2}, \ldots, d_u\}$, and $\{d_{u+1}, d_{u+2}, \ldots, d_k\}$ so that for all $1 \leqslant i \leqslant k$, $d_i \in \{d_1, d_2, \ldots, d_r\}$ if $d_{i,\Phi(d_i)} = 0$ and $|d_i| = \lceil k/2 \rceil + 1$, $d_i \in \{d_{r+1}, d_{r+2}, \ldots, d_u\}$ if $d_{i,\Phi(d_i)} = 0$, $|d_i| = \lceil k/2 \rceil$, and $\overline{d_i} \notin \{d_1, d_2, \ldots, d_k\}$, and $d_i \in \{d_{u+1}, d_{u+2}, \ldots, d_k\}$ else, where $0 \leqslant r \leqslant u \leqslant k$. Initially, let $D = \{\overline{d_1}, \ldots, \overline{d_u}, d_{u+1}, \ldots, d_k\}$ and define a minimal routing function $\Psi : D \to \{1, 2, \ldots, k\}$ as follows: $\Psi(\overline{d_j}) = \Phi(d_j)$ for all $1 \leqslant j \leqslant u$ and $\Psi(d_t) = \Phi(d_t)$ for all $u+1 \leqslant t \leqslant k$. Whenever there is a $d_l \in D$ so that $u+1 \leqslant l \leqslant k$, $d_{l,\Psi(d_l)} = 0$, $|d_l| = \lceil k/2 \rceil$, and $\overline{d_l} \in D$, D and Ψ need to be modified as follows. Let $p_l = p_{l,1}, p_{l,2}, \ldots, p_{l,k}$ be an adjacent node of $\overline{d_l}$ so that $|p_l| = |\overline{d_l}| - 1$, $p_l \notin D$, and $\overline{p_l} \notin D \cup \{d_{k+1}\}$, and assume $\overline{d_l} = d_h$ for some $u+1 \leqslant h \leqslant k$ and $h \neq l$. First, swap $\Psi(d_l)$ and $\Psi(d_h)$ if $p_{l,\Psi(d_l)} = 0$. Then, replace d_l with p_l in D, and assign $\Psi(p_l)$ with $\Psi(d_l)$. Without loss of generality, assume $D = \{\overline{d_1}, \ldots, \overline{d_u}, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_k\}$ finally, where $u \leqslant v \leqslant k$. */
(62) If $\overline{d_i} = d_w$ for some $v+1 \leqslant w \leqslant k$, then {
(67) If $\overline{d_i} \neq d_j$ for all $v+1 \leqslant j \leqslant k$, then
(74) /* Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_{u'}\}$ and $\{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ so that for all $1 \leqslant i \leqslant k$, $d_i \in \{d_1, d_2, \ldots, d_{u'}\}$ if $|d_i| \geqslant \lceil k/2 \rceil + 2$, and $d_i \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ else, where $0 \leqslant u' \leqslant k$. Initially, let $D = \{\overline{d_1}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ and determine a minimal routing function $\Omega : D \to \{1, 2, \ldots, k\}$. Whenever there is a $\overline{d_h} \in D$, $1 \leqslant h \leqslant u'$, so that $|\overline{d_h}| = \lfloor k/2 \rfloor - 2$, $\overline{d_h} = d_l$ for some $u'+1 \leqslant l \leqslant k$, and $(\overline{d_h}_{,\Omega(\overline{d_h})} = 0$ or $\overline{d_h}_{,\Omega(d_l)} = 0)$, D and Ω need to be modified as follows. First, swap $\Omega(\overline{d_h})$ and $\Omega(d_l)$ if $\overline{d_h}_{,\Omega(\overline{d_h})} = 1$. Let $q_h = q_{h,1} q_{h,2}, \ldots, q_{h,k}$ be an adjacent node of $\overline{d_h}$ so that $q_h \notin D$, $\overline{q_h} \notin D$, and for some $w \in \{1, 2, \ldots, k\} - \{\Omega(z) | z = z_1 z_2 \ldots z_k \in D, z_{\Omega(z)} = 1$, and $|z| \leqslant \lfloor k/2 \rfloor - 1\}$, $q_{h,w} = 1$, $\overline{d_h}_{,w} = 0$, and $q_{h,i} = \overline{d_h}_{,i}$ for all $1 \leqslant i \leqslant k$ and $i \neq w$. Then, replace $\overline{d_h}$ with q_h in D, and determine a new minimal routing function Ω arbitrarily. Without loss of generality, assume $D = \{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ finally, where $0 \leqslant r' \leqslant u'$. */
(81) If $\overline{d_j} = d_t$ for some $u'+1 \leqslant t \leqslant k$, then {
(86) If $\overline{d_j} \neq d_v$ for all $u'+1 \leqslant v \leqslant k$, then.

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